# 6 Numerical methods

## 6.1 The intermediate value theorem

The **intermediate value theorem** states that if a continuous function $f$ with an interval $[a, b]$ as its domain takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.

Draw a diagram to illustrate the intermediate value theorem:

It has an important corollary, called **Bolzano's theorem**: if a continuous function $f(x)$ has values of opposite signs at the ends of an interval $[a, b]$, then it has (at least) a root in that interval.

Draw a diagram to illustrate Bolzano's theorem:

**Exercise 41.**

1. Show that each of the following functions has at least one root in its given interval.

$$\begin{array}{ll} y = x + \cos(\pi x); & [-1, 0] \\ y = x^2 - \mathrm{e}^{-x}; & [0, 1] \\ y = x^3 - 2x^2 - x + 1; & [0, 1] \\ y = \ln(x^2 + 1) - 2 + x; & [1, 2] \end{array}$$

2. Find three integer values of $N$ such that the equation $x^4 = 2^x$ has a root in the interval $[N, N + 1]$.

3. Show that the equation $\sqrt[3]{x^2 + 1} = x$ has a root in the interval $[1, 2]$. Then use numerical methods to find this root, correct to 3 decimal places. You may sketch a diagram to illustrate the question.

4. Use numerical methods to find the solution, between 7 and 8, of the equation: $\tan x = x$, correct to 3 decimal places. You may sketch a diagram to illustrate the question.

## 6.2 Solving equations by iteration

When solving an equation $F(x) = 0$, it is sometimes useful to rearrange to the form

$$x = f(x),$$

which is called an **iterative formula**. A root, $\alpha$ of the equation $F(x) = 0$, is called a **fixed point**, or an **invariant point**, or an **equilibrium** of the function $f(x)$, as $f(\alpha) = \alpha$.

Starting with a number $x_0$ (sufficiently) close to $\alpha$, ($x_0 \approx \alpha$), we may apply the iterative formula to produce a sequence of numbers:

$$x_0,\ x_1 = f(x_0),\ x_2 = f(x_1),\ \cdots,\ x_{k+1} = f(x_k),\ \cdots.$$

One possibility is that this sequence may approach $\alpha$ as $n \to \infty$:

$$\lim_{n\to\infty} x_k = \alpha.$$

In this case, $\alpha$ is called a **stable** fixed point, or a **stable** equilibrium, or an **attractor**. We may use this iterative formula to find the approximate value of $\alpha$.

Try to find a root of the equation $\sqrt[3]{x+1} = x$, by starting with $x_0 = 1$, correct to 3 decimal places. You may draw a diagram to illustrate how this iteration proceeds.

**Exercise 42.**

1. Use the iteration $x = \tan^{-1}(x^2 + 1)$ to find the root of the equation $\tan x = x^2 + 1$ correct to 2 decimal places, showing the result of each iteration to 4 decimal places.
2. Show that the equation $2^x = x^3$ has a root between 1 and 2.
   Use an iterative method based on the rearrangement $x = 2^{\frac{x}{3}}$, with initial approximation $x_0 = 2$, to find the value of the root correct to 2 decimal places, showing the result of each iteration to 4 decimal places.
3. Show that the equation $\mathrm{e}^{\sqrt{x}} = x^2 - 1$ has a root between 2 and 3.
   Use an iterative method based on the rearrangement $x = \sqrt{\mathrm{e}^{\sqrt{x}} + 1}$, with initial approximation $x_0 = 2$, to find the value of the root correct to 2 decimal places, showing the result of each iteration to 4 decimal places.
4. Sketch the graphs of $y = x$ and $y = \cos x$, and state the number of roots of the equation $x = \cos x$.
   Use a suitable iteration and starting point to find this root, correct to 3 decimal places.
   What if we use the iteration $x = \cos^{-1} x$ starting from $x = 0$?
5. Show that the equation $2^x + 3^x = 6^x$ has a root between 1.5 and 2.
   Use a suitable iteration to solve this equation, correct to 2 decimal places.

The other possibility is that, even if the starting point of iteration, $x_0$, is picked very close to $\alpha$, the sequence $\{x_k\}$ still fails to converge to $\alpha$, instead it may tend to infinity, converge to some other value, or even oscillates. In this case, $\alpha$ is called an **unstable** fixed point, or an **unstable** equilibrium, or a **repeller**.

For each of the following example, draw diagrams to illustrate why the iterations fail.

1. The equation $x^3 = x$ has root $\alpha = 1$.
   The iteration with initial value $x_0 = 0.9$ tends to another root 0,
   while the iteration with initial value $x_0 = 1.1$ tends to infinity.

2. The equation $1.3 \sin x = x$ has root $\alpha = 0$.
   Any iteration with a positive initial value, no matter how small, tends to the positive root around 1.2215.
   Similarly, any iteration with a negative initial value tends to the negative root around $-1.2215$.

(‡) Now we look at the conditions of convergence of such iterations.

In fact, for a fixed point $\alpha$ of a function $f(x)$, if $|f'(\alpha)| < 1$ then attraction is guaranteed.

A slightly looser condition is **Lipschitz continuity** with Lipschitz constant $L < 1$ near the fixed point $\alpha$, that is:

$$\forall s, t \approx \alpha, \quad |f(s) - f(t)| \leq L|s - t|.$$

**Exercise 43.**

1. It is given that the equation $x^3 - x - 1 = 0$ has only one real root $\alpha$.
   (a) Show that $1 < \alpha < 2$.
   (b) Explain why the iteration $x = x^3 - 1$ fails to converge to $\alpha$.
   (c) Explain why the iteration $x = \sqrt[3]{x+1}$ works, and use this iteration to give an estimate value of $\alpha$ correct to 2 decimal places.

2. (†) The equation is given to be
   $$e^x = x^2 + 5x - 1 \qquad (*)$$
   (a) By carefully sketching both $y = \mathrm{e}^x$ and $y = x^2 + 5x - 1$, determine the number of roots of $(*)$.
   (b) Use the iterative formula $x = \ln(x^2 + 5x - 1)$ to find the root between 3 and 4, correct to 2 decimal places.
   (c) Try to work out two other iterations to find the other roots, correct to 2 decimal places.

## 6.3 The trapezium rule

The **trapezium rule** is used to estimate e definite integral. When the integrating interval $[a, b]$ is equally divided into $n$ sub-intervals $[x_{k-1}, x_k]$ $(k = 1, 2, 3 \ldots, n)$, where $x_k = a + kh$, and $h = \dfrac{b-a}{n}$ is the width of each sub-interval, the integral is estimated as:

$$\int_a^b f(x)\,\mathrm{d}x \approx h\left\{\tfrac{1}{2}f(x_0) + [f(x_1) + f(x_2) + \cdots + f(x_{n-1})] + \tfrac{1}{2}f(x_n)\right\}.$$

Use the trapezium rule with 4 intervals to estimate the value of $\displaystyle\int_2^6 \ln x\,\mathrm{d}x$.

Draw a diagram to illustrate whether this is an overestimate or an underestimate.

From this example, you can see that the estimation given by the trapezium rule is an over- respective underestimate,

if the integrand $f(x)$ is a __________ respective __________ function on the interval $[a, b]$;

in another word, if __________ respective __________, for $x \in [a, b]$.

**Exercise 44.**

1. Use the trapezium rule with 5 intervals to estimate the value of $\displaystyle\int_{\frac{1}{2}}^{3} \left(x + \frac{1}{x}\right)\mathrm{d}x$.

2. Use the trapezium rule with 6 intervals to estimate the value of $\displaystyle\int_0^{\pi} \sin x\,\mathrm{d}x$, correct to 2 decimal places. Compare with the true value of the integral and explain why this is an underestimate.

3. Use the trapezium rule with (a) 4 intervals; (b) 6 intervals; (c) 8 intervals, to estimate the value of $\displaystyle\int_0^1 \frac{1}{x^2+1}\,\mathrm{d}x$, each correct to 4 decimal places. Compare the results with the true value of the integral.

4. Use the trapezium rule with 5 intervals to estimate the value of $\displaystyle\int_1^6 \mathrm{e}^{\frac{1}{x}}\,\mathrm{d}x$, correct to 3 decimal places. Determine whether this is an overestimate or an underestimate.